

Jointly optimised piecewise-linear kernels and a certified lower bound $C_{1a} \geq 1.30758$ for the supremum of autoconvolutions

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Abstract Let C_{1a} be the largest constant with $\|f * f\|_\infty \geq C_{1a}(\int f)^2$ for every nonnegative f supported in $[-\frac{1}{4}, \frac{1}{4}]$. We prove $C_{1a} \geq 65379/50000 = 1.30758$. The best refereed lower bound we are aware of is 1.28 (Cloninger–Steinerberger, 2017), and a 2026 public repository claims 1.292. The proof uses the Fourier-analytic method of Martin–O’Bryant and Matolcsi–Vinuesa without change. What is new is the choice of its two free functions: a class of continuous piecewise-linear kernels on a grid commensurate with the period, for which positivity of the kernel’s Fourier transform at all relevant frequencies reduces to finitely many exact checks; a joint convex optimisation of kernel and multiplier; and a kernel support $\delta \approx 0.21$, about 1.5 times wider than in earlier work. The certificate (a kernel with 240 and a multiplier with 800 rational coefficients) is verified in exact rational and ball arithmetic in about one minute, was re-verified by an independent implementation in exact integer arithmetic, and the final algebraic deduction is formalised in Lean 4. Diagnostics indicate that this method saturates near 1.308; further progress needs a new idea.

Keywords Autoconvolution · Sidon sets · Fourier analysis · Computer-assisted proof · Convex optimisation · Interval arithmetic

1 Introduction

For nonnegative $f \in L^1(\mathbb{R})$ supported in $[-\frac{1}{4}, \frac{1}{4}]$ with $\int f = 1$ the autoconvolution $f * f$ is supported in $[-\frac{1}{2}, \frac{1}{2}]$ and has integral 1, so $\|f * f\|_\infty \geq 1$. The constant

$$C_{1a} = \inf \left\{ \|f * f\|_\infty : f \geq 0, \text{supp } f \subset [-\frac{1}{4}, \frac{1}{4}], \int f = 1 \right\} \quad (1)$$

measures how far $f * f$ must exceed its average. It governs the size of finite sets with few repeated sums (generalised Sidon sets) [4, 3] and is entry 1a of the public table of optimisation constants maintained by Tao and collaborators [7].

Known bounds. Martin and O’Bryant [4] introduced a Fourier-analytic lower-bound method; Matolcsi and Vinuesa [5] sharpened it with an explicit kernel and multiplier and obtained $C_{1a} \geq 1.2748$. Cloninger and Steinerberger [3] proved $C_{1a} \geq 1.28$ by a different, discretisation-based computer-assisted argument. In 2026 Piterbarg, Bajaj and Vincent claimed $C_{1a} \geq 1.292$ in a public repository with a written proof and partial Lean formalisation [6]; we have not checked that work. On the other side, step-function constructions give upper bounds; the smallest value we are aware of is 1.50274 (a 65 536-step function with an exact rational scorer, 2026) [2]. The gap between 1.29 and 1.50 is wide.

Summary of results.

- Theorem 2: $C_{1a} \geq 65379/50000 = 1.30758$.
- Theorem 1 restates the master inequality of [4, 5] for a general kernel, with a complete proof; Lemma 1 shows that for piecewise-linear kernels on a grid of step u/m its positivity hypothesis is equivalent to $\lfloor m/2 \rfloor + 1$ inequalities.
- The kernel and the multiplier are optimised jointly by a convex program (Section 4). Two changes matter: optimising the kernel instead of fixing it, and widening its support from $\delta \approx 0.138$ to $\delta = 30/143 \approx 0.2098$ (Table 1).
- The certificate is checked rigorously, re-checked independently, attacked with negative controls, and its final algebraic step is formalised in Lean 4 (Section 5).
- The method saturates: for our kernel class the attainable bound grows from 1.3054 ($N = 200$) to 1.30759 ($N = 800$) with shrinking increments, and a primal computation confines the best bound for the certified kernel to $[1.307584, 1.307787]$ (Section 6).

2 The master inequality

Throughout, $f \geq 0$ is supported in $[-\frac{1}{4}, \frac{1}{4}]$ with $\int f = 1$, $M = \|f * f\|_\infty$, and $f \circ f(x) = \int f(t)f(t+x)dt$ is the auto-correlation. Both $f * f$ and $f \circ f$ are supported in $[-\frac{1}{2}, \frac{1}{2}]$ and have integral 1; in particular $M \geq 1$.

Definition 1. Let $0 < \delta < \frac{1}{2}$ and $u = \frac{1}{2} + \delta$. A *kernel* is an even $K \geq 0$ in $L^2(\mathbb{R})$ with $\text{supp } K \subset [-\delta, \delta]$ and $\int K = 1$; we write $K_2 = \int K^2$ and $\hat{K}(\xi) = \int K(x)e^{-2\pi i x \xi} dx$. A *multiplier* is $G(x) = \sum_{j=1}^N a_j \cos(2\pi j x/u)$ with real a_j . The pair is *admissible* if $\hat{K}(j/u) \geq 0$ for all $j \in \mathbb{Z}$, $\hat{K}(j/u) > 0$ whenever $a_j \neq 0$, and $G \geq m_G > 0$ on $[0, \frac{1}{4}]$.

Theorem 1 (Master inequality, [4, 5]). *For an admissible pair and every f as above,*

$$M + 1 + \sqrt{(M-1)(K_2-1)} \geq \tau := \frac{2}{u} + \frac{4}{u} \frac{m_G^2}{S_1}, \quad (2)$$

where $S_1 = \sum_{a_j \neq 0} a_j^2 / \hat{K}(j/u)$.

Proof. Step 1. Since $K \geq 0$ and $\int K = 1$, $\int (f * f)K \leq M$.

Step 2 (period 1). Let $\hat{f}(n) = \int f(x)e^{-2\pi i n x} dx$. The functions $f \circ f$, $f * f$ (supported in an interval of length 1) and K (supported in $[-\delta, \delta]$, $\delta < \frac{1}{2}$) agree with their

1-periodisations on $[-\frac{1}{2}, \frac{1}{2}]$ up to null sets, with Fourier coefficients $|\hat{f}(n)|^2$, $\hat{f}(n)^2$ and $\hat{K}(n)$. Parseval and Cauchy–Schwarz give

$$\begin{aligned} \int (f \circ f)K &= 1 + \sum_{n \neq 0} |\hat{f}(n)|^2 \hat{K}(n) \\ &\leq 1 + \left(\sum_{n \neq 0} |\hat{f}(n)|^4 \right)^{1/2} \left(\sum_{n \neq 0} \hat{K}(n)^2 \right)^{1/2}. \end{aligned}$$

Here $\sum_{n \neq 0} |\hat{f}(n)|^4 = \int (f * f)^2 - 1 \leq M \int f * f - 1 = M - 1$ and $\sum_{n \neq 0} \hat{K}(n)^2 = K_2 - 1$. Hence $\int (f \circ f)K \leq 1 + \sqrt{(M-1)(K_2-1)}$.

Step 3 (period u). Let $c_j = \int f(x) e^{-2\pi i j x / u} dx$, so $c_0 = 1$ and $c_{-j} = \overline{c_j}$. For $|x| \leq \delta$ and $k \neq 0$ we have $|x - ku| \geq u - \delta = \frac{1}{2}$, so on $[-\delta, \delta]$ the u -periodic convolution and correlation of the u -periodisation of f coincide with $f * f$ and $f \circ f$. Since also $\delta < u/2$, Parseval for period u gives

$$\begin{aligned} \int (f * f + f \circ f)K &= \frac{1}{u} \sum_{j \in \mathbb{Z}} (c_j^2 + |c_j|^2) \hat{K}(j/u) \\ &= \frac{2}{u} + \frac{4}{u} \sum_{j \geq 1} (\operatorname{Re} c_j)^2 \hat{K}(j/u), \end{aligned}$$

using $c^2 + |c|^2 = 2(\operatorname{Re} c)^2 + 2i \operatorname{Re} c \operatorname{Im} c$, evenness of $\hat{K}$ and $c_{-j} = \overline{c_j}$.

Step 4. Drop the terms with $a_j = 0$ (they are ≥ 0) and apply Cauchy–Schwarz:

$$\sum_{j \geq 1} (\operatorname{Re} c_j)^2 \hat{K}(j/u) \geq \frac{(\sum_j a_j \operatorname{Re} c_j)^2}{S_1} = \frac{(\int f G)^2}{S_1} \geq \frac{m_G^2}{S_1},$$

because G is even, $G \geq m_G > 0$ on $[-\frac{1}{4}, \frac{1}{4}]$ and $\int f = 1$. Combining Steps 1–4 gives (2). $\square$

Corollary 1. *If an admissible pair and a number $M_t \geq 1$ satisfy $\Phi(M_t) := M_t + 1 + \sqrt{(M_t - 1)(K_2 - 1)} < \tau$, then $C_{1a} > M_t$.*

Proof. Φ is increasing on $[1, \infty)$ and $\Phi(M) \geq \tau > \Phi(M_t)$ for every admissible f by Theorem 1. $\square$

Remark 1. [4, 5] state the method for $\delta \leq \frac{1}{4}$; the proof above only uses $\delta < \frac{1}{2}$. Our certificate has $\delta = 30/143 < \frac{1}{4}$, so it lies within their stated range. The kernel of [5] is the autoconvolution of an arcsine density; Theorem 1 holds for every kernel in the sense of Definition 1, as the proof shows.

3 Piecewise-linear kernels

Fix integers $K_n \geq 0$ and $m > 2(K_n + 1)$. Put $\delta = \frac{K_n + 1}{2(m - K_n - 1)}$, $u = \frac{1}{2} + \delta$ and $h = u/m$; then $(K_n + 1)h = \delta$. Let $\Lambda(t) = \max(0, 1 - |t|)$ and, for $c_0, \dots, c_{K_n} \geq 0$,

$$K(x) = \sum_{|k| \leq K_n} c_{|k|} \Lambda(x/h - k). \quad (3)$$

Lemma 1. *K is even, nonnegative, supported in $[-\delta, \delta]$, $\int K = h(c_0 + 2\sum_{k \geq 1} c_k)$ and*

$$\begin{aligned} \hat{K}(j/u) &= h \operatorname{sinc}^2(j/m) P_j, \\ P_j &= c_0 + 2 \sum_{k=1}^{K_n} c_k \cos(2\pi k j / m), \end{aligned}$$

Table 1 Effect of the kernel support δ (floating point, not certified). Kernels are nonnegative mixtures of Matolcsi–Vinuesa kernels (autoconvolutions of arcsine densities) of half-widths up to δ , optimised jointly with G , $N = 200$.

δ	0.10	0.138	0.16	0.18	0.20	0.25
bound	1.2726	1.2941	1.3001	1.3032	1.3042	1.2963

Table 2 Piecewise-linear kernels at $\delta = 30/143$: optimum (floating point) and certified bound.

N	K_n	m	optimum	certified
200	59	203	1.30540	1.30519
400	119	406	1.30683	1.30672
600	179	609	1.30734	–
800	239	812	1.30759	1.30758

with $\operatorname{sinc} t = \sin(\pi t)/(\pi t)$. Since P_j depends only on $j \bmod m$ and $P_{-j} = P_j$, the condition $\hat{K}(j/u) \geq 0$ for all $j \in \mathbb{Z}$ is equivalent to $P_r \geq 0$ for $r = 0, \dots, \lfloor m/2 \rfloor$. Moreover, with node values $v_s = K(sh)$,

$$K_2 = \frac{h}{3} \sum_s (v_s^2 + v_s v_{s+1} + v_{s+1}^2),$$

a rational number if h and all c_k are rational.

Proof. $\hat{\Lambda}(\xi) = \operatorname{sinc}^2 \xi$, so $\hat{K}(\xi) = h \operatorname{sinc}^2(h\xi) \sum_{|k| \leq K_n} c_{|k|} e^{-2\pi i k h \xi}$, and $h j / u = j / m$. The formula for K_2 integrates the square of a linear function on each interval $[sh, (s+1)h]$. $\square$

Requiring $N < m$ ensures $\operatorname{sinc}(j/m) \neq 0$ for all $1 \leq j \leq N$, so admissibility reduces to $P_j > 0$ for those j with $a_j \neq 0$ (and $P_r \geq 0$ for all residues). The kernel of [5] has a singular peak at 0; our optimised kernels develop a similar sharp peak (Fig. 1a) but are otherwise free.

4 Joint optimisation

For fixed (K_n, m, N) and a cap κ on K_2 we solve

$$\begin{aligned} \min_{a, c} \quad & \sum_{j=1}^N \frac{a_j^2}{h \operatorname{sinc}^2(j/m) P_j(c)} \\ \text{s.t.} \quad & G(x_i) \geq 1 \text{ on a grid } x_i \in [0, \tfrac{1}{4}], \quad c \geq 0, \quad \int K = 1, \\ & P_r(c) \geq 0 \quad (0 \leq r \leq \lfloor m/2 \rfloor), \quad K_2(c) \leq \kappa. \end{aligned}$$

Each summand is a quadratic-over-linear function and hence jointly convex in (a_j, c) ; the constraints are linear or convex quadratic. We solve it with CVXPY and Clarabel and obtain M from $\Phi(M) = \tau$ with $m_G = 1$, then scan κ . The optimum is flat in κ near $K_2 \approx 3.8$.

Table 1 separates the two improvements over [5]. At the support $\delta \approx 0.138$ used there, optimising a mixture kernel jointly with G already gives 1.294 instead of 1.2748; widening the support to $\delta \approx 0.2$ adds another 0.01. Piecewise-linear kernels at $\delta = 30/143$ add a further 0.003 (Table 2). As a calibration of our formulas, inserting the 119 coefficients printed in [5] reproduces the gain 0.0713 reported there.

5 The certificate

Theorem 2. $C_{1a} \geq 65379/50000 = 1.30758$.

Proof. The data file `cert_final.json` contains $K_n = 239$, $m = 812$ (so $\delta = 30/143$, $u = 203/286$), rational kernel values $c_0, \dots, c_{239}$ and rational multiplier coefficients $a_1, \dots, a_{800}$, obtained by rounding the optimiser’s output to denominators 10^9 and renormalising $\int K = 1$ exactly. The checker `certify_pl.py` verifies, with exact rationals (Python `fractions`) and Arb ball arithmetic [1] at 160 bits:

- (A) $c_k \geq 0$ and $\int K = 1$ exactly; $0 < \delta < \frac{1}{2}$;
- (B) $P_r > 0$ for $r = 0, \dots, 406$ (minimum $P_r \geq 3.556$), hence admissibility by Lemma 1;
- (C) $K_2 = 3.8000000019\dots$ exactly (rational, 24-digit denominator);
- (D) $S_1 \leq 13.48415970779$;
- (E) $G \geq m_G = 199987/200000$ on $[0, \frac{1}{4}]$, by a branch-and-bound with the second-order bound $G(x) \geq G(x_0) - |G'(x_0)|r - \frac{1}{2}L_2r^2$, $L_2 \geq \sup |G''|$, on 32351 cells;
- (F) $\tau \geq 3.2356121164365$ and $\Phi(M_t) \leq 3.2356015517897$ for $M_t = 65379/50000$.

Corollary 1 gives $C_{1a} > M_t$. The run takes about one minute on one core. $\square$

5.1 Independent validation

Second implementation. A separate checker, written without access to our modules, recomputes all conditions in exact integer and rational arithmetic, with π from Machin’s formula and integer Taylor enclosures of $\cos$ and $\sin$ (cross-checked against `mpmath.iv` on 5612 angles); it evaluates $\hat{K}(j/u)$ in two independent ways. It confirms (A)–(F): $K_2 = 3.800000001905818$, $S_1 \leq 13.48415970778676$, $\min_{[0, 1/4]} G \geq 0.9999450$ (with a rigorous floating-point error bound; the true minimum is 0.9999452 at $x \approx 0.2491$), and it accepts $M_t = 1.307584$ while rejecting 1.307585.

Negative controls. The checker must reject: a target $M_t = 1.307585$ (fails (F)); a kernel with mass $13/25$ or another perturbed mass (fails (A)); a kernel with a negative Fourier value (fails (B)); a too high m_G (fails (E)); a negative m_G (rejected since version 2, see Remark 2). All seven controls behave as required.

Lean formalisation. The deduction in Corollary 1, with the certified enclosures $\tau \geq \tau_0$ and $K_2 \leq K_{2,hi}$, is formalised in Lean 4 with Mathlib (`PayneCore/Autoconv.lean`, no sorry): for every linearly ordered field, $s \geq 0$, $s^2 \leq (M-1)(K_2-1)$, $\tau \leq M+1+s$, $\tau \geq 3.2356121164365$, $1 \leq K_2 \leq 3.80000000191$ imply $M > 65379/50000$. The analytic Steps 1–4 and the floating-point-free numerical checks (A)–(F) are not formalised.

Remark 2 (Status of the computer-assisted parts). All pass/fail decisions use exact rationals or Arb balls; no floating-point number enters them. An adversarial audit of version 1 of the checker found two defects that do not affect this certificate but could have certified false bounds for other inputs: a negative m_G was not rejected, and the

hypothesis checks were Python `assert` statements, which python -0 skips. Version 2 rejects $m_G \leq 0$ and uses explicit checks; Theorem 2 was re-certified with it (SHA-256 of the certificate `fd9ee69d\dots`).

6 How much is left in this method?

Saturation in N . At fixed $\delta = 30/143$ the optimum grows by 0.00143, 0.00051 and 0.00025 from $N = 200$ to 400, 600 and 800 (Table 2). Extrapolating these decreasing increments suggests a limit near 1.308 for this kernel class.

Primal check for the certified kernel. Minimising $\int (f * f + f \circ f)K$ directly over step functions on a fine grid gives 3.2361200, slightly above $\tau = 3.2356121$. Hence even an optimal multiplier for this kernel cannot push the bound above $\Phi^{-1}(3.2361200) \approx 1.307787$, and the certified 1.30758 is within $2.1 \cdot 10^{-4}$ of that ceiling.

Where the slack is. The Fourier side (Steps 3–4) is nearly tight, as the primal check shows. The loss sits in Step 2, where Cauchy–Schwarz bounds $\int (f \circ f)K$ through $\int (f * f)^2 \leq M$ and K_2 without using the joint structure of $f * f$ and $f \circ f$.

What did not help. Matolcsi and Vinuesa also treat the first Fourier coefficient $z_1 = |\hat{f}(1)|$ separately and combine several kernels through $\min_{z_1} \max_K$ of the resulting bounds [5]. With our kernels this improved the bound by only about $2 \cdot 10^{-4}$, so we did not include it in the certificate.

7 Discussion

The improvement from 1.2748 [5] to 1.30758 does not come from a new inequality but from treating the kernel as an optimisation variable and from a wider support. Lemma 1 is what makes this practical: on a grid commensurate with the period, Fourier positivity at infinitely many frequencies becomes a finite, exactly checkable condition, so the optimiser can search over all kernels of a given resolution rather than a one-parameter family.

Our bound exceeds the claimed 1.292 of [6] and the refereed 1.28 of [3]. It has not been refereed; the certificate, both checkers and the Lean file are available for checking.

Open questions.

1. Is there a replacement for Step 2 that uses the relation between $f * f$ and $f \circ f$ (for example through $\int (f * f)(f \circ f)$ or a second kernel)? This is where the method loses most.
2. What is the supremum of the right-hand side of (2) over all admissible pairs? Our numerics suggest about 1.308.
3. Can the remaining gap to the upper bound 1.5027 [2] be narrowed from above by structured (non-step) constructions?

Declarations

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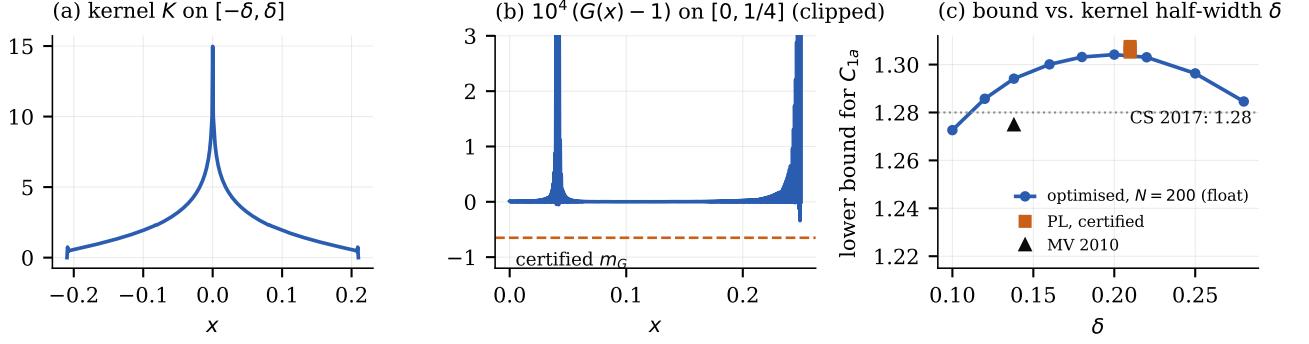


Fig. 1 (a) The certified piecewise-linear kernel. (b) The multiplier minus one on $[0, \frac{1}{4}]$ (scale 10^{-4} , spikes clipped) and the certified lower bound m_G . (c) Lower bound versus kernel half-width δ (Table 1), the certified piecewise-linear bounds of Table 2, the bound of [5] and the refereed record 1.28 [3].

Code and data availability. Certificate `cert_final.json` (SHA-256 `fd9ee69d34c150df8f5254261b7e0d85b9c3a7cfd5e9ad1f4692f227e2243f6`), checker `certify_pl.py` (version 2), optimiser `opt_pl.py`, the independent checker and the Lean project are provided with this preprint. Reproduction: `python3 -I certify_pl.py cert_final.json`.

Competing interests. None.

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A Numerical details

Rounding. Optimiser output is rounded to rationals with denominator 10^9 ; c_0 is then adjusted so that $\int K = 1$ holds exactly. The target m_G is set slightly below the grid minimum of G and the target M_t slightly below the floating-point optimum, so that (E) and (F) hold with margins $1.0 \cdot 10^{-5}$ (in m_G) and $1.06 \cdot 10^{-5}$ (in $\tau - \Phi$).

Branch-and-bound for G . On a cell $[x_0 - r, x_0 + r]$ the checker evaluates $G(x_0)$ and $G'(x_0)$ as Arb balls and uses

$L_2 = \sum_j |a_j| (2\pi j/u)^2$ as an upper bound for $|G''|$. Cells failing the bound are bisected; failure below width 10^{-12} rejects the certificate.

Software. Python 3, python-flint (Arb), CVXPY with Clarabel, mpmath; Lean 4 (v4.35.0-rc3) with Mathlib.